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Explicit and Implicit Continuation Algorithms for Strongly Monotone Variational Inequalities with Box Constraints

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Abstract. In this paper we discuss the variational inequality problems VIP(X,F), where F is assumed to be a strongly monotone mapping from $\Re^n$ to $\Re^n$, and the feasible set $X=[l,u]$ has the form of box constraints. Based on the Chen-Harker-Kanzow smoothing functions, first we present an explicit continuation algorithm (ECA) for solving VIP(X,F). The ECA possesses main features as follows: (a) at each iteration, it yields a new iterative point by solving a system of equations in $\Re^{(n+s)}$ with a parameter and nonsingular Jacobian matrix, where $s=|\{j: -\infty<l_j<u_j<+\infty\}|$, (b) it generates a sequence of iterative points in the interior of the feasible set X. Secondly we give an implicit continuation algorithm (ICA) for solving VIP(X,F), the prime character of the ICA is that it solves only one, rather than a series of, system of nonlinear equations to obtain a solution of VIP(X,F). The two proposed algorithms are shown to possess strongly global convergence. Finally, some preliminary numerical results of the two algorithms are reported.

Mathematics Subject Classifications. 90C30, 65K05.

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1. Introduction

This paper concerns the solution of the following variational inequality problem (VIP). Let

$$l=(l_1,\ldots,l_n)^T,\ u=(u_1,\ldots,u_n)^T,\ l_i\in\Re\cup\{-\infty\},\ u_i\in\Re\cup\{+\infty\},\ l_i\neq u_i,$$
$$F(x)=(F_1(x),\ldots,F_n(x))^T:\Re^n\mapsto\Re^n.$$

Then VIP with box constraints is to find a vector $x^*\in\Re^n$ such that

$$\text{VIP}(X,F)\quad x^*\in X,\qquad F(x^*)^T(x-x^*)\geqslant 0,\quad \forall x\in X\overset{\text{def}}{=}[l,u]. \tag{1.1}$$

This problem has extensive applications. For example, VIP(X,F) of the form (1.1) can be considered as special cases of the standard variational inequality problem

$$\text{VIP}(C,F)\quad x^*\in C,\qquad F(x^*)^T(x-x^*)\geqslant 0,\quad \forall x\in C\subseteq\Re^n, \tag{1.2}$$

and the variational inequality problem with inequality constraints

$$\text{VIP}(D,F) \quad x^* \in D, \qquad F(x^*)^T(x-x^*) \geqslant 0,$$
$$\forall x \in D \overset{\text{def}}{=} \{x \in \Re^n \,|\, g_i(x) \geqslant 0,\ i \in \mathcal{A}\}, \tag{1.3}$$

where the index set $\mathcal{A} = \{1, \ldots, \kappa\}$. In addition, the nonlinear complementarity problem (NCP)

$$x \geqslant 0,\ F(x) \geqslant 0,\ F(x)^T x = 0, \tag{1.4}$$

and the normal bound constraints VIP can be regarded as special cases of problem (1.1) if we take $l = 0, u = +\infty$ and $l, u \in \Re^n$ respectively. If $F(x)$ is a gradient function of some real-value function $f: \Re^n \mapsto \Re$, then the problem (1.1) is equivalent to the stationary condition of optimization problem $\min\{f(x) \,|\, l \leqslant x \leqslant u\}$.

It is known that the optimizing methods and the continuation methods have recently become two kinds of very important and effective approaches to solving VIPs and NCPs. The basic idea of the former is to transform a VIP or/and a NCP into an equivalent (in a sense) optimization problem, and that of the latter is to reformulate a VIP or/and a NCP as an equivalent system of nonlinear equations. For example, under mild conditions, Kanzow and Fukushima [11] and Ferris et al. [4] used respectively a so called D-gap function and a complementarity function (CP-function) to transform the box constrained VIP(X, F) into an equivalent unconstrained optimization problem and an optimization problem with the simple box constrained set X, then they presented the associated algorithms based on the equivalent programs; Hotta and Yoshise [6], with the help of the Chen-Harker-Kanzow-Smale CP-function, used a homotopy function and optimization technique to present an effective algorithm for solving the standard NCP under mild conditions.

The continuation methods for VIP(D, F) (1.3) are based generally on a known result as follows: if all functions g_i are concave and the linearly independent constrained qualification for the feasible set D of (1.3) holds, Harker and Pang [5] proved that the problem (1.3) is equivalent to the following KKT problem

$$F(x) - \sum_{i \in \mathcal{A}} y_i \nabla g_i(x) = 0,\ g_i(x) \geqslant 0,\ y_i \geqslant 0,\ y_i g_i(x) = 0, \quad \forall i \in \mathcal{A}. \tag{1.5}$$

Based on problem (1.5) above and a generalized complementarity function (GCP-function), the continuation methods transform the VIP (1.3) into an equivalent system of nonlinear equations, see Refs. [1, 7, 8, 12]. Our interest in this paper is laid on the continuation method for the box constrained VIP (1.1). Although the problem (1.1) discussed in the paper may be solved theoretically by the proposed continuation methods [1, 7, 8, 12], the number of the multiplier variables y_i in (1.5) would be $n+s$, where $s = |\{i \,|\, -\infty < l_i < u_i < +\infty\}|$, and the system of equations solved at each iteration would consist of $2n+s$ equations and $2n+s$

variables, so the scale would be very large. Hence the proposed continuation methods (see [1, 7, 8, 12]) for VIP (1.3) would be ineffective if they were used directly to solve the problem (1.1).

Based on the reasons above, this paper presents directly an explicit continuation algorithms and an implicit continuation algorithms for the problem (1.1) with box constraints. The main ideas of the algorithms are, by means of the Chen-Harker-Kanzow function, to transform respectively the problem (1.1) into an equivalent sequence of system of nonlinear equations which consists of only $n+s$ equations and $n+s$ variables, and only one equivalent system of nonlinear equations.

The structure of this paper is as follows. In Section 2, the explicit continuation algorithm (ECA for abbreviation) is given and its some important properties are discussed. Section 3 proves the existence and uniqueness of the solution for the system of equations needed to be solved in the ECA. Section 4 analyses and proves the strongly global convergence and the stability of the ECA. The implicit continuation algorithm (ICA for abbreviation) is given in Section 5. Some preliminary numerical results are reported in Section 6. We conclude with some final remarks in Section 7.

For sets J and I, we use the following notation throughout this paper:

$$x_J=(x_j,\ j\in J),\ F_J(x)=(F_j(x),\ j\in J),$$
$$\nabla_{x_J}F_I(x)=\left(a_{ji}=\frac{\partial F_i(x)}{\partial x_j},\ j\in J,\ i\in I\right), \tag{1.6}$$

that is $\nabla_{x_J}F_I(x)$ denotes the gradient (matrix) of vector value function $F_I(x)$ with respect to x_J, so the transpose $(\nabla_{x_J}F_I(x))^T$ denotes the Jacobian matrix of function $F_I(x)$ with respect to x_J.

2. The Explicit Continuation Algorithm

We first recall some well-known definitions and results which will be used in this paper.

DEFINITION. A function $F: C\to\Re^n$ is said to be:

(i) monotone over set C if

$$\left(F(x^1)-F(x^2)\right)^T(x^1-x^2)\geqslant 0,\quad \forall x^1,x^2\in C;$$

(ii) strongly monotone over set C (with modulus $\alpha>0$) if

$$\left(F(x^1)-F(x^2)\right)^T(x^1-x^2)\geqslant \alpha\|x^1-x^2\|^2,\quad \forall x^1,x^2\in C.$$

THEOREM 1. *Suppose that $C\subseteq\Re^n$ is a nonempty, closed and convex set, and $F: C\to\Re^n$ is a strongly monotone and continuous function. Then the problem* VIP(C,F) (1.2) *has a unique solution.*

THEOREM 2. *Suppose that the function F is continuous and functions g_i are concave and continuously differentiable, and the linear independence constraint qualification* (*LICQ, see Definition* 2 *in* [8]) *holds for the feasible set D in problem* (1.3). *Then the problem* VIP(D,F) (1.3) *is equivalent to the KKT problem* (1.5), *i.e.,* (x,y) *is a solution of* (1.5) *if and only if x is a solution of* (1.3).

Theorem 1 above can be seen in [5] (Corollary 3.2) or [12] (Theorem 2.2) or [9] (Theorem 2.12), and Theorem 2 above can be seen in [5] (Proposition 2.2).

Throughout this paper, we suppose the following assumption holds.

ASSUMPTION A. The function $F:\Re^n \mapsto \Re^n$ in (1.1) is continuously differentiable and strongly monotone.

For convenience, we divide the set $\{1,\dots,n\}$ into four subsets as follows:

$$I=\{i\,|\,-\infty<l_i<u_i=+\infty\},\quad J=\{j\,|\,-\infty=l_j<u_j<+\infty\},$$
$$P=\{p\,|\,-\infty<l_p<u_p<+\infty\},\quad Q=\{q\,|\,-\infty=l_q,\ u_q=+\infty\}.$$

Without loss of generality, furthermore suppose that

$$I=\{1,\dots,m\},\quad J=\{m+1,\dots,m+r\},$$
$$P=\{m+r+1,\dots,m+r+s\},\quad Q=\{m+r+s+1,\dots,(m+r+s+h)=n\},$$

and denote vector y by

$$y=(y_p,\ p\in P)\in\Re^s.$$

THEOREM 3. *The problem* (1.1) *and the following system* (2.1):

$$(x_i-l_i)F_i(x)=0,\ x_i-l_i\geqslant 0,\ F_i(x)\geqslant 0,\quad \forall i\in I, \tag{2.1a}$$
$$-F_j(x)(u_j-x_j)=0,\ u_j-x_j\geqslant 0,\ -F_j(x)\geqslant 0,\quad \forall j\in J, \tag{2.1b}$$
$$\big(F_p(x)+y_p\big)(x_p-l_p)=0,\ F_p(x)+y_p\geqslant 0,\ x_p-l_p\geqslant 0,\quad \forall p\in P, \tag{2.1c}$$
$$F_q(x)=0,\quad \forall q\in Q, \tag{2.1d}$$
$$y_p(u_p-x_p)=0,\ y_p\geqslant 0,\ u_p-x_p\geqslant 0,\quad \forall p\in P, \tag{2.1e}$$

are equivalent, i.e., x is a solution of (1.1) *if and only if there exists a $y\in\Re^s$ such that (x,y) is a solution of* (2.1). *Moreover, both the problems* (1.1) *and* (2.1) *have a unique solution.*

Proof. To finish the proof by Theorem 2, we define

$$g_i(x)=x_i-l_i, i\in I;\; g_j(x)=u_j-x_j, j\in J;\; g_p(x)=x_p-l_p, p\in P;$$
$$g_{p+s}(x)=u_p-x_p, p\in P,\; e_j=(0,\dots,0,1(j\text{th}),0,\dots,0)^T\in\Re^n, j=1,\dots,n. \tag{2.2}$$

Then (1.1) is a special case of (1.3) where functions g_i are defined by formula (2.2) above and the index set $\mathcal{A}=\{1,2,\dots,m+r+2s\}$. Since the LICQ for the feasible set $X=D$ always holds and functions g_i are all concave and continuously differentiable, thus we can conclude from Theorem 2 that the problem (1.1) is equivalent to the problem (1.5), i.e., the problem (1.1) is equivalent to the following problem:

$$F(x)-\sum_{i\in I}y_ie_i+\sum_{j\in J}y_je_j-\sum_{p\in P}y'_pe_p+\sum_{p\in P}y_pe_p=0, \tag{2.3a}$$

$$y_i(x_i-l_i)=0,\; y_i\geqslant 0,\quad x_i-l_i\geqslant 0,\; i\in I;$$
$$y_j(u_j-x_j)=0,\; y_j\geqslant 0,\quad u_j-x_j\geqslant 0,\; j\in J; \tag{2.3b}$$
$$y'_p(x_p-l_p)=0,\; y'_p\geqslant 0,\quad x_p-l_p\geqslant 0,\; p\in P;$$
$$y_p(u_p-x_p)=0,\; y_p\geqslant 0,\quad u_p-x_p\geqslant 0,\; p\in P. \tag{2.3c}$$

On the other hand, it is obvious that the problem (2.3) is equivalent to (2.1), so the equivalency between (1.1) and (2.1) is proved.

Finally, according to the fact that X is a closed convex set and F is strongly monotone and Theorem 1, we know that (1.1), and so (2.1), has a unique solution. So the proof is finished. □

Let parameter $\mu\geqslant 0$, consider the following perturbed complementarity problem associated with (2.1):

$$(x_i-l_i)F_i(x)=\mu,\; x_i-l_i\geqslant 0,\; F_i(x)\geqslant 0,\quad \forall i\in I, \tag{2.4a}$$
$$-F_j(x)(u_j-x_j)=\mu,\; u_j-x_j\geqslant 0,\; -F_j(x)\geqslant 0,\quad \forall j\in J, \tag{2.4b}$$
$$\big(F_p(x)+y_p\big)(x_p-l_p)=\mu,\; F_p(x)+y_p\geqslant 0,\; x_p-l_p\geqslant 0,\quad \forall p\in P, \tag{2.4c}$$
$$F_q(x)=0,\quad \forall q\in Q, \tag{2.4d}$$
$$y_p(u_p-x_p)=\mu,\; y_p\geqslant 0,\; u_p-x_p\geqslant 0,\quad \forall p\in P. \tag{2.4e}$$

We know, with the help of some generalized complementarity function (GCP-function) $\phi\colon \Re^3\to\Re$ (see [2, 8, 10, 13]), that problem (2.4) can be reformulated equivalently as a system of nonlinear equations. In this paper, we choose the GCP-function given by Chen, Harker and Kanzow [2, 11] as follows.

$$\phi(a,b,\mu)=a+b-\sqrt{(a-b)^2+4\mu},\quad \text{for } (a,b,\mu)\in\Re^2\times[0,+\infty). \tag{2.5}$$

Of course, one may choose other forms of GCP-function (see Section 7 of [10]), and they have the same role. The following results on function ϕ can be proved easily or seen in [13].

LEMMA 1. *For any* $\mu \geqslant 0$, *we have*

(i) $\phi(a,b,\mu)=0$ *if and only if* $a \geqslant 0, b \geqslant 0, ab=\mu$;
(ii) $\phi(a,b,\mu)=c$ *if and only if* $(a-c/2) \geqslant 0, (b-c/2) \geqslant 0$ *and* $(a-c/2)(b-c/2)=\mu$;
(iii) $\phi(a,b,\mu) \geqslant 0$ *if and only if* $a \geqslant 0, b \geqslant 0, ab \geqslant \mu$.

Let us define vector-value functions by

$$\Phi_I(x,y,\mu)=(\phi(x_i-l_i, F_i(x),\mu),\ i \in I), \tag{2.6a}$$

$$\Phi_J(x,y,\mu)=(-\phi(u_j-x_j, -F_j(x),\mu),\ j \in J), \tag{2.6b}$$

$$\Phi_P^1(x,y,\mu)=(\phi(x_p-l_p, F_p(x)+y_p,\mu),\ p \in P), \tag{2.6c}$$

$$\Phi_Q(x,y,\mu)=F_Q(x)=(F_q(x),\ q \in Q), \tag{2.6d}$$

$$\Phi_P^2(x,y,\mu)=(-\phi(u_p-x_p, y_p,\mu),\ p \in P), \tag{2.6e}$$

$$\Phi(x,y,\mu)=\begin{pmatrix}\Phi_I(x,y,\mu)\\ \Phi_J(x,y,\mu)\\ \Phi_P^1(x,y,\mu)\\ \Phi_Q(x,y,\mu)\\ \Phi_P^2(x,y,\mu)\end{pmatrix}. \tag{2.7}$$

From Lemma 1, one has directly the following result.

THEOREM 4. *For any* $\mu \geqslant 0$, *the equation system* $\Phi(x,y,\mu)=0$ *and the system* (2.4) *are completely equivalent, i.e.,* (x,y,μ) *is a solution of* $\Phi(x,y,\mu)=0$ *if and only if it is a solution of* (2.4).

The reasons why we use the minus sign for $i \in J \cup P$ in (2.6b) and (2.6e) are follows. The minus sign for $j \in J$ in (2.6b) can ensure the nonsingularity of the Jacobian matrix of Φ, and the following proposition motivates why the minus sign for $p \in P$ in (2.6e) is used.

PROPOSITION 1. *Let* $(x,y) \in \Re^n \times \Re^s$ *be fixed, and* $\mu \geqslant 0$. *Then the following hold*:

(i) $\lim_{a \to +\infty} \phi(a,b,\mu)=2b,\ \lim_{b \to +\infty} \phi(a,b,\mu)=2a$;
(ii) $2\gamma \overset{\text{def}}{=} \lim_{l_p \to -\infty} \phi(x_p-l_p, F_p(x)+y_p,\mu)=2(F_p(x)+y_p)$;

(iii) $-\phi(u_p - x_p, y_p, \mu) = -\phi(u_p - x_p, \gamma - F_p(x), \mu)$, *which has the similar form of* (2.6*b*) *if* γ *small enough.*

Proof. We have from (2.5)

$$\begin{aligned}\lim_{a\to+\infty}\phi(a,b,\mu) &= \lim_{a\to+\infty}\left((a+b)-\sqrt{(a-b)^2+4\mu}\right)\\ &= \lim_{a\to+\infty}\frac{\left((a+b)-\sqrt{(a-b)^2+4\mu}\right)\left((a+b)+\sqrt{(a-b)^2+4\mu}\right)}{a+b+\sqrt{(a-b)^2+4\mu}}\\ &= \lim_{a\to+\infty}\frac{4ab-4\mu}{a+b+\sqrt{(a-b)^2+4\mu}}\\ &= \lim_{a\to+\infty}\frac{4b-4\frac{\mu}{a}}{1+\frac{b}{a}+\sqrt{\left(1-\frac{b}{a}\right)^2+4\frac{\mu}{a^2}}} = 2b.\end{aligned}$$

Similarly, one has $\lim_{b\to+\infty}\phi(a,b,\mu) = 2a$. Moreover, we have from part (i)

$$\lim_{l_p\to-\infty}\phi(x_p-l_p, F_p(x)+y_p, \mu) = \lim_{a\to+\infty}\phi(a, F_p(x)+y_p, \mu) = 2(F_p(x)+y_p).$$

Lastly, in view of $y_p = \gamma - F_p(x)$, part (iii) is clear and the proof is completed. □

For convenience of discussion, we denote the partial derivatives of ϕ by

$$\begin{aligned}\psi(a,b,\mu) &\overset{\text{def}}{=} \frac{\partial\phi(a,b,\mu)}{\partial a} = 1-\frac{a-b}{\sqrt{(a-b)^2+4\mu}},\\ \theta(a,b,\mu) &\overset{\text{def}}{=} \frac{\partial\phi(a,b,\mu)}{\partial b} = 1+\frac{a-b}{\sqrt{(a-b)^2+4\mu}}.\end{aligned} \tag{2.8}$$

The functions ψ and θ above possess an important property as follows, its proof is elementary and omitted.

PROPOSITION 2. *The functions* $\psi(\cdot)$ *and* $\theta(\cdot)$ *satisfy*

$$0 < \psi(a,b,\mu) < 2,\ 0 < \theta(a,b,\mu) < 2,\quad \forall(a,b,\mu)\in\Re^2\times(0,+\infty). \tag{2.9}$$

To analyse the gradient matrix of function $\Phi(x,y,\mu)$, we introduce the vectors and diagonal matrices as follows:

$$\begin{aligned}&d=(x,y)\quad z=(x,y,\mu),\\ &D_I = \text{diag}(\psi(x_i-l_i, F_i(x), \mu),\ i\in I),\\ &R_I = \text{diag}(\theta(x_i-l_i, F_i(x), \mu),\ i\in I),\end{aligned} \tag{2.10}$$

$$D_J = \mathrm{diag}(\psi(u_j - x_j, -F_j(x), \mu), j \in J),$$

$$R_J = \mathrm{diag}(\theta(u_j - x_j, -F_j(x), \mu), j \in J), \tag{2.11}$$

$$D_P^1 = \mathrm{diag}(\psi(x_p - l_p, F_p(x) + y_p, \mu), p \in P),$$

$$R_P^1 = \mathrm{diag}(\theta(x_p - l_p, F_p(x) + y_p, \mu), p \in P), \tag{2.12}$$

$$D_P^2 = \mathrm{diag}(\psi(u_p - x_p, y_p, \mu),\ p \in P),$$

$$R_P^2 = \mathrm{diag}(\theta(u_p - x_p, y_p, \mu),\ p \in P), \tag{2.13}$$

$$D = \begin{pmatrix} D_I & & & \\ & D_J & & \\ & & D_P^1 & \\ & & & 0_{h\times h} \end{pmatrix}, \quad R = \begin{pmatrix} R_I & & & \\ & R_J & & \\ & & R_P^1 & \\ & & & I_{h\times h} \end{pmatrix}, \tag{2.14}$$

$$H = (0_{s\times m},\ 0_{s\times r},\ R_P^1,\ 0_{s\times h}), \quad G^T = (0_{s\times m},\ 0_{s\times r},\ D_P^2,\ 0_{s\times h}), \tag{2.15}$$

By elementary computation and analysis, the gradients of functions $\Phi_I, \Phi_J, \Phi_P^1, \Phi_P^2$ and Φ_Q can be given by the following proposition.

PROPOSITION 3. *For any parameter* $\mu > 0$, *the functions* $\Phi_I(x, y, \mu)$, $\Phi_J(x, y, \mu)$, $\Phi_P^1(x, y, \mu)$, $\Phi_P^2(x, y, \mu)$ *and* $\Phi_Q(x, y, \mu)$ *are all continuously differentiable on* $\Re^n \times \Re^s$, *and their gradients can be expressed as follows, denoted them simply by* $\nabla_d \Phi_I$, $\nabla_d \Phi_J$, $\nabla_d \Phi_P^1$, $\nabla_d \Phi_Q$, *and* $\nabla_d \Phi_P^2$, *respectively.*

$$\nabla_d \Phi_I = \begin{pmatrix} D_I + \nabla_{x_I} F_I(x) R_I \\ \nabla_{x_J} F_I(x) R_I \\ \nabla_{x_P} F_I(x) R_I \\ \nabla_{x_Q} F_I(x) R_I \\ 0_{s\times m} \end{pmatrix}, \quad \nabla_d \Phi_J = \begin{pmatrix} \nabla_{x_I} F_J(x) R_J \\ D_J + \nabla_{x_J} F_J(x) R_J \\ \nabla_{x_P} F_J(x) R_J \\ \nabla_{x_Q} F_J(x) R_J \\ 0_{s\times r} \end{pmatrix}, \tag{2.16}$$

$$\nabla_d \Phi_P^1 = \begin{pmatrix} \nabla_{x_I} F_P(x) R_P^1 \\ \nabla_{x_J} F_P(x) R_P^1 \\ D_P^1 + \nabla_{x_P} F_P(x) R_P^1 \\ \nabla_{x_Q} F_P(x) R_P^1 \\ R_P^1 \end{pmatrix}, \quad \nabla_d \Phi_Q = \begin{pmatrix} \nabla_{x_I} F_Q(x) \\ \nabla_{x_J} F_Q(x) \\ \nabla_{x_P} F_Q(x) \\ \nabla_{x_Q} F_Q(x) \\ 0_{s\times h} \end{pmatrix},$$

$$\nabla_d \Phi_P^2 = \begin{pmatrix} 0_{m\times s} \\ 0_{r\times s} \\ D_P^2 \\ 0_{h\times s} \\ -R_P^2 \end{pmatrix}, \tag{2.17}$$

$$\begin{aligned} \nabla_d \Phi(x, y, \mu) &= (\nabla_d \Phi_I, \nabla_d \Phi_J, \nabla_d \Phi_P^1, \nabla_d \Phi_Q, \nabla_d \Phi_P^2) \\ &= \begin{pmatrix} D + \nabla F(x) R & G \\ H & -R_P^2 \end{pmatrix}. \end{aligned} \tag{2.18}$$

THEOREM 5. *The gradients* $\nabla_d \Phi(x, y, \mu)$, *and so the Jacobian matrices* $(\nabla_d \Phi(x, y, \mu))^T$, *of* $\Phi(x, y, \mu)$ *are nonsingular for all* $d = (x, y) \in \Re^n \times \Re^s$ *and all* $\mu > 0$.

Proof. It is sufficient to show that the equation system $\nabla_d \Phi(x, y, \mu)(w^T, v^T)^T = 0$ has a unique solution zero. Suppose that $(w^T, v^T)^T \in \Re^{(n+s)}$ such that $\nabla_d \Phi(x, y, \mu)(w^T, v^T)^T = 0$, then we have from (2.18)

$$\nabla_d \Phi(x, y, \mu) \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} Dw + \nabla F(x) Rw + Gv \\ Hw - R_P^2 v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Combining this equations with (2.15), we deduce

$$(Rw)^T (Dw + \nabla F(x) Rw + Gv) = 0, \quad Hw - R_P^2 v = R_P^1 w_P - R_P^2 v = 0. \tag{2.19}$$

From (2.14)–(2.15) and the second equation of (2.19), we obtain

$$w^T RGv = (R_P^1 w_P)^T D_P^2 v = (R_P^2 v)^T D_P^2 v = v^T (R_P^2 D_P^2) v. \tag{2.20}$$

which, together with the first equation of (2.19), gives

$$w^T (RD) w + (Rw)^T \nabla F(x)(Rw) + v^T (R_P^2 D_P^2) v = 0. \tag{2.21}$$

In addition, Proposition 2 (i.e., (2.9)) shows that all the matrices D_I, D_J, D_P^1, D_P^2, R_I, R_J, R_P^1 and R_P^2 are all diagonal and positive definite, furthermore, the matrix D (see (2.14)) is diagonal and positive semi-definite, and the matrices R and $R_P^2 D_P^2$ both are diagonal and positive definite. Also, the matrix $\nabla F(x)$ is positive definite for $x \in \Re^n$ since $F(x)$ is strongly monotone (see Theorem 2.8 in [9]). Thus, from (2.21), we have $Rw = 0$ and $v = 0$, which show that $(w, v) = (0, 0)$. Hence the proof is completed. □

Based on the system of equations $\Phi(x, y, \mu) = 0$, now we present our explicit continuation algorithm for problem (1.1) as follows.

Explicit continuation algorithm (ECA for abbreviation)

Step 0. *Choose a stopping tolerance* $\delta > 0$ *and an error function* $\mathrm{err}(x, y)$ *(its specific construction can be seen in the later formula* (2.22) *in this paper or in* [1]*), choose an arbitrary initial point* $x^o \in X = [l, u], y^o \in \Re^s$*, and any sequence* $\{\mu_k\}$ *such that* $\mu_k > 0$ *and* $\lim_{k \to \infty} \mu_k = 0$. *Let* $k = 0$*, go to Step* 1*;*

Step 1. *Starting with* (x^k, y^k)*, solve the equation system* $\Phi(x, y, \mu_{k+1}) = 0$ *by some given method (e.g. Newton, Newton type or quasi-Newton methods) to obtain a new point* (x^{k+1}, y^{k+1})*, i.e., the (approximate) solution of* $\Phi(x, y, \mu_{k+1}) = 0$*;*

Step 2. *If* $err(x^{k+1}, y^{k+1}) < \delta$, *stop. Otherwise, let* $k := k+1$, *go back to Step* 1.

Based on formula (2.1), we can construct a specific error function $\mathrm{err}(x, y)$ as follows.

$$\begin{aligned}\mathrm{err}(x,y) = &\sum_{i\in I}(\min\{x_i - l_i, F_i(x)\})^2 + \sum_{j\in J}(\min\{u_j - x_j, -F_j(x)\})^2 + \\ &+ \|F_Q(x)\|^2 + \sum_{p\in P}\{(\min\{y_p, u_p - x_p\})^2 + \\ &+ (\min\{x_p - l_p, F_p(x) + y_p\})^2\}.\end{aligned} \tag{2.22}$$

THEOREM 6 (SEE [17]). *Let* $\mu > 0$ *and* (x^μ, y^μ) *be a solution of* $\Phi(x, y, \mu) = 0$. *Suppose that Newton method is used to solve* $\Phi(x, y, \mu) = 0$ *with initial point* (x^o, y^o) *located in a small neighbourhood of* (x^μ, y^μ). *Then the ECA will converge to* (x^μ, y^μ) *at a quadratic rate.*

3. Existence and Uniqueness of the Solution to Equation $\Phi(x, y, \mu) = 0$

In this section, we prove that the equation $\Phi(x, y, \mu) = 0$ has a unique solution and it is continuous with respect to the parameter $\mu > 0$. The following lemma from [14] is useful in the subsequent proof.

LEMMA 2. *Suppose that* $a_k, b_k \geqslant 0,\ k = 1, 2$. *Then*

$$(a_1 - b_1)(a_2 - b_2) \leqslant |a_1 a_2 - b_1 b_2|. \tag{3.1}$$

LEMMA 3. *Let* $\mu_1 > 0,\ \mu_2 > 0$, *and suppose that* (x^1, y^1) *and* (x^2, y^2) *are solutions of* $\Phi(x, y, \mu_1) = 0$ *and* $\Phi(x, y, \mu_2) = 0$, *respectively. Then*

$$\alpha\|x^1 - x^2\|^2 \leqslant (F(x^1) - F(x^2))^T(x^1 - x^2) \leqslant (m + r + 2s)|\mu_1 - \mu_2|. \tag{3.2}$$

Proof. From Theorem 4 one knows (x^1, y^1) and (x^2, y^2) are solutions of system (2.4) for parameters μ_1 and μ_2, respectively. So we have from (2.4*a*) (note that $x_i^1 - l_i > 0$, $x_i^2 - l_i > 0$, $i \in I$)

$$F_i(x^1) = \frac{\mu_1}{x_i^1 - l_i}, \qquad F_i(x^2) = \frac{\mu_2}{x_i^2 - l_i}, \quad i \in I.$$

Multiplying the two equations above by $(x_i^1 - x_i^2)$ and $(x_i^2 - x_i^1)$ respectively, then adding them, we have

$$(F_i(x^1) - F_i(x^2))(x_i^1 - x_i^2) = (x_i^1 - x_i^2)\left(\frac{\mu_1}{x_i^1 - l_i} - \frac{\mu_2}{x_i^2 - l_i}\right), \quad i \in I.$$

On the other hand, we get from Lemma 2

$$(x_i^1 - x_i^2)\left(\frac{\mu_1}{x_i^1 - l_i} - \frac{\mu_2}{x_i^2 - l_i}\right) = ((x_i^1 - l_i) - (x_i^2 - l_i))\left(\frac{\mu_1}{x_i^1 - l_i} - \frac{\mu_2}{x_i^2 - l_i}\right) \leqslant |\mu_1 - \mu_2|.$$

Thus

$$(F_i(x^1) - F_i(x^2))(x_i^1 - x_i^2) \leqslant |\mu_1 - \mu_2|, \quad i \in I.$$

$$(F_I(x^1) - F_I(x^2))^T (x_I^1 - x_I^2) \leqslant |I| \cdot |\mu_1 - \mu_2| = m|\mu_1 - \mu_2|. \tag{3.3}$$

Similarly, we can prove

$$(F_J(x^1) - F_J(x^2))^T (x_J^1 - x_J^2) \leqslant r|\mu_1 - \mu_2|. \tag{3.4}$$

Also in view of $x_p^1 - l_p > 0, x_p^2 - l_p > 0$, we have from (2.4$c$)

$$F_p(x^1) = \frac{\mu_1}{x_p^1 - l_p} - y_p^1, \qquad F_p(x^2) = \frac{\mu_2}{x_p^2 - l_p} - y_p^2, \quad p \in P.$$

Multiplying the two equations above by $(x_p^1 - x_p^2)$ and $(x_p^2 - x_p^1)$ respectively, then adding them, we have

$$\begin{aligned}(F_p(x^1) - F_p(x^2))(x_p^1 - x_p^2) &= (x_p^1 - x_p^2)\left(\frac{\mu_1}{x_p^1 - l_p} - \frac{\mu_2}{x_p^2 - l_p}\right) + (x_p^2 - x_p^1)(y_p^1 - y_p^2) \\ &= ((x_p^1 - l_p) - (x_p^2 - l_p))\left(\frac{\mu_1}{x_p^1 - l_p} - \frac{\mu_2}{x_p^2 - l_p}\right) + \\ &\quad + ((u_p - x_p^1) - (u_p - x_p^2))(y_p^1 - y_p^2).\end{aligned}$$

We also have from (2.4e)

$$y_p^1 - y_p^2 = \frac{\mu_1}{u_p - x_p^1} - \frac{\mu_2}{u_p - x_p^2}.$$

Hence we know from Lemma 2

$$((x_p^1 - l_p) - (x_p^2 - l_p))\left(\frac{\mu_1}{x_p^1 - l_p} - \frac{\mu_2}{x_p^2 - l_p}\right) \leqslant |\mu_1 - \mu_2|,$$

$$\begin{aligned}((u_p - x_p^1) - (u_p - x_p^2))(y_p^1 - y_p^2) &= ((u_p - x_p^1) - (u_p - x_p^2)) \times \\ &\quad \times \left(\frac{\mu_1}{u_p - x_p^1} - \frac{\mu_2}{u_p - x_p^2}\right) \leqslant |\mu_1 - \mu_2|.\end{aligned}$$

So we have

$$(F_p(x^1)-F_p(x^2))(x_p^1-x_p^2)\leqslant|\mu_1-\mu_2|+((u_p-x_p^1)-\\ -(u_p-x_p^2))(y_p^1-y_p^2),\quad p\in P. \tag{3.5}$$

$$(F_p(x^1)-F_p(x^2))(x_p^1-x_p^2)\leqslant 2|\mu_1-\mu_2|,\quad p\in P.$$

$$(F_P(x^1)-F_P(x^2))^T(x_P^1-x_P^2)\leqslant 2|P|\cdot|\mu_1-\mu_2|=2s|\mu_1-\mu_2|. \tag{3.6}$$

On the other hand, we show from $(2.4d)$

$$(F_Q(x^1)-F_Q(x^2))^T(x_Q^1-x_Q^2)=0. \tag{3.7}$$

Thus combining (3.3), (3.4), (3.6), (3.8) and Assumption A, we can conclude (3.2) holds. So the proof is completed. □

THEOREM 7. *The equation system* $\Phi(x,y,\mu)=0$*, i.e., the system* (2.4) *has at most one solution for all* $\mu>0$*. Furthermore, the solution is continuous with respect to the parameter* μ*.*

The proof is obvious from formula (3.2) in Lemma 3.

THEOREM 8. *The equation system* $\Phi(x,y,\mu)=0$ *has a unique solution for all* $\mu>0$*.*

Proof. In view of Theorem 7, it is sufficient to show the existence. In order to use the known results in [12] to predigest and complete the proof, we consider the functions given by (2.2). Similar to the proof of Theorem 3, it is not difficult to verify that the system (2.4) and the following perturbed nonlinear complementarity problem are completely equivalent, denoted by PVIP(X,F,μ):

$$F(x)-\sum_{i\in I\cup J}y_i\nabla g_i(x)-\sum_{p\in P}y_p\nabla g_p(x)-\sum_{p\in P}y_{p+s}\nabla g_{p+s}(x)=0,$$

$$y_ig_i(x)=\mu,\ y_i\geqslant 0,\ g_i(x)\geqslant 0,\ i\in I\cup J,$$

$$y_pg_p(x)=\mu,\ y_p\geqslant 0,\ g_p(x)\geqslant 0,\ p\in P,$$

$$y_{p+s}g_{p+s}(x)=\mu,\ y_{p+s}\geqslant 0,\ g_{p+s}(x)\geqslant 0,\ p\in P.$$

Since LICQ always holds at any point $x\in\Re^n$ and F is assumed to be strongly monotone, and notice that the problem VIP(X,F) has unique solution (Theorem 1), we can conclude, from Theorem 3.15 in [12], that the problem PVIP(X,F,μ) given above has a solution for all $\mu>0$. So the system (2.4) has a solution, moreover, we conclude from Theorem 4 that $\Phi(x,y,\mu)=0$ has a solution for all $\mu>0$. The proof is finished. □

4. The Strong Convergence of the ECA

In Section 3, we have studied the existence, uniqueness and continuity of the solution of $\Phi(x,y,\mu)=0$. In this section, we will prove the solution (x^k,y^k) of $\Phi(x,y,\mu_k)=0$ approaches to the unique solution (x^*,y^*) of the system (2.1), and x^k converges to the unique solution of (1.1).

LEMMA 4. *Let* (x^μ,y^μ) *be the unique solution of the equation system* $\Phi(x,y,\mu)=0$. *If a set* $\Omega\subset\Re_+=\{t\in\Re\,|\,t>0\}$ *is bounded, then the solution set* $\{(x^\mu,y^\mu)\,|\,\mu\in\Omega\}$ *is also bounded.*

Proof. From the boundedness of Ω, without loss of generality, we suppose $\mu\leqslant\beta,\forall\mu\in\Omega$. Let $\bar\mu>0$ be a fixed parameter, we know from Theorem 8 that $\Phi(x,y,\bar\mu)=0$ has a unique solution $(\bar x,\bar y)$. Moreover, we have from (3.2) for any $\mu\in\Omega$

$$\alpha\|x^\mu-\bar x\|^2\leqslant(m+r+2s)|\mu-\bar\mu|\leqslant(m+r+2s)(\beta+\bar\mu).$$

This shows that $\{x^\mu\,|\,\mu\in\Omega\}$ is bounded.

Next, we analyse the boundedness of $\{y^\mu\,|\,\mu\in\Omega\}$. One has from (3.5)

$$(F_P(x^\mu)-F_P(\bar x))^T(x_P^\mu-\bar x_P)\leqslant s|\mu-\bar\mu|+((u_P-x_P^\mu)-(u_P-\bar x_P))^T(y^\mu-\bar y).$$

On the other hand, one knows from $(2.4e)$

$$u_P-\bar x_P>0,\ \bar y>0,\ u_P-x_P^\mu>0,\ (u_P-\bar x_P)^T\bar y=s\bar\mu,\ (u_P-x_P^\mu)^Ty^\mu=s\mu.$$

So we have

$$\begin{aligned}((u_P-x_P^\mu)-(u_P-\bar x_P))^T(y^\mu-\bar y)&=(u_P-x_P^\mu)^Ty^\mu+(u_P-\bar x_P)^T\bar y-\\&\quad-(u_P-x_P^\mu)^T\bar y-(u_P-\bar x_P)^Ty^\mu\\&=s(\mu+\bar\mu)-(u_P-x_P^\mu)^T\bar y-(u_P-\bar x_P)^Ty^\mu\\&\leqslant s(\mu+\bar\mu)-(u_P-\bar x_P)^Ty^\mu,\end{aligned}$$

$$\begin{aligned}(F_P(x^\mu)-F_P(\bar x))^T(x_P^\mu-\bar x_P)&\leqslant 2s(\mu+\bar\mu)-(u_P-\bar x_P)^Ty^\mu\\&\leqslant 2s(\beta+\bar\mu)-(u_P-\bar x_P)^Ty^\mu\end{aligned}$$

Since $\{x^\mu\,|\,\mu\in\Omega\}$ has been proved to be bounded and $F(x)$ is continuous, there exists a constant $M>0$ such that

$$\|(F_P(x^\mu)-F_P(\bar x))^T(x_P^\mu-\bar x_P)\|\leqslant M,\quad\forall\mu\in\Omega.$$

Thus

$$(u_P-\bar x_P)^Ty^\mu\leqslant-(F_P(x^\mu)-F_P(\bar x))^T(x_P^\mu-\bar x_P)+2s(\beta+\bar\mu)\leqslant 2s(\beta+\bar\mu)+M.$$

This inequality and $(u_P-\bar{x}_P,\ y^{\mu})>(0,0)$ show that the set $\{y^{\mu}\,|\,\mu\in\Omega\}$ is bounded. Thus the proof of Lemma 4 has been finished. □

THEOREM 9. *Suppose that the parameter sequence* $\{\mu_k\}$ *chosen in ECA is arbitrary such that* $\mu_k>0$ *and* $\mu_k\to 0$. *Then the entire sequence* $\{(x^k,y^k)\}$ *generated by ECA converges to the unique solution* (x^*,y^*) *of the system* (2.1), *therefore,* $\{x^k\}$ *converges to the unique solution* x^* *of problem* (1.1), *that is ECA is strongly convergent.*

Proof. Firstly, from Lemma 4 and $\mu_k\to 0$, we know that $\{(x^k,y^k)\}$ is bounded, so it has at least one limit point, and let $(\hat{x},\hat{y})$ be any given accumulation point. Secondly, since (x^k,y^k) is a solution of $\Phi(x,y,\mu_k)=0$, i.e., the system (2.4) for $\mu=\mu_k$, it is easy to verify $(\hat{x},\hat{y})$ is a solution of the system (2.1). Finally, in view of the uniqueness of the solution of (2.1), we can conclude $(\hat{x},\hat{y})=(x^*,y^*)$. Thus $\{(x^k,y^k)\}$ has a unique accmulation point (x^*,y^*), therefore the entire sequence $\{(x^k,y^k)\}$ converges to the solution (x^*,y^*) of the system (2.1) by the boundedness of $\{(x^k,y^k)\}$. Furthermore, from Theorem 3, we conclude that the entire sequence $\{x^k\}$ converges to the solution x^* of problem (1.1). □

Theorem 9 indicates the ECA possesses satisfactory convergence. However, in order to analyse further its numerical stability, we need to discuss further the properties of the gradient $\nabla_d\Phi(x^*,y^*,0)$. For this goal, the following additional assumption is necessary.

ASSUMPTION B. Suppose the strict complementarity conditions hold at the solution x^* of problem (1.1), i.e.,

$$(x_i^*-l_i,F_i(x^*))\neq(0,0),\quad \forall i\in I;\quad (u_j^*-x_j^*,-F_j(x^*))\neq(0,0),\quad \forall j\in J;$$

$$F_p(x^*)>0,\quad \forall p\in P_l\overset{\text{def}}{=}\{p\in P:\ x_p^*-l_p=0\};$$

$$F_p(x^*)<0,\quad \forall p\in P_u\overset{\text{def}}{=}\{p\in P:\ u_p-x_p^*=0\}.$$

It is obvious, for the solution (x^*,y^*) of problem (2.1), that Assumption B and the following conditions are equivalent:

$$(x_i^*-l_i,F_i(x^*))\neq(0,0),\quad \forall i\in I;\quad (u_j-x_j^*,-F_j(x^*))\neq(0,0),\quad \forall j\in J;$$

$$(x_p^*-l_p,F_p(x^*)+y_p^*)\neq(0,0),\quad (u_p-x_p^*,y_p^*)\neq(0,0),\quad \forall p\in P.$$

THEOREM 10. *Suppose that Assumptions A and B hold, then the function* $\Phi(x,y,0)$ *is continuously differentiable at the solution* (x^*,y^*) *of* (2.1), *and the gradient* $\nabla_d\Phi(x^*,y^*,0)$, *so the Jacobian matrix* $(\nabla_d\Phi(x^*,y^*,0))^T$, *is nonsingular. Furthermore there exists a constant* $c>0$ *such that*

$$\|\nabla_d\Phi(x^k,y^k,\mu_k)^{-1}\|\leqslant c,\quad \textit{for all sufficiently large } k.$$

Proof. By combining (2.8), (2.10)–(2.15), (2.18) as well as Assumption B, we can conclude that the function $\Phi(x,y,0)$ is continuously differentiable at point (x^*,y^*) and $\nabla_d\Phi(x^*,y^*,0)$ has the same formula as (2.18). To complete the rest of the proof, it is sufficient to verify the equation system $\nabla_d\Phi(x^*,y^*,0)(w^T,v^T)^T=0$ has a unique solution zero. Suppose that $(w^T,v^T)^T=(w_I^T,w_J^T,w_P^T,w_Q^T,v_P^T)^T\in\Re^{(n+s)}$ such that $\nabla_d\Phi(x^*,y^*,0)(w^T,v^T)^T=0$, then we have from (2.18)

$$\nabla_d\Phi(x^*,y^*,0)\begin{pmatrix}w\\ v\end{pmatrix}=\begin{pmatrix}D^*w+\nabla F(x^*)R^*w+G^*v\\ H^*w-R_P^{*2}v\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{4.1}$$

where the matrices D^*,R^*,G^*,H^*,R_P^{*2} and so on are those defined by (2.10)–(2.15) corresponding to $(x^*,y^*,0)$. So we obtain from (4.1) and (2.15)

$$(R^*w)^T(D^*w+\nabla F(x^*)R^*w+G^*v)=0,\ H^*w-R_P^{*2}v=R_P^{*1}w_P-R_P^{*2}v=0. \tag{4.2}$$

On the other hand, from formulas (2.8), (2.10)–(2.13), Assumption B and taking into account (x^*,y^*) being a solution of (2.1), we know

$$(D_I^*)_{ii}=\psi(x_i^*-l_i,F_i(x^*),0)=\begin{cases}2, & i\in I_l\overset{\text{def}}{=}\{i\in I:\ x_i^*-l_i=0\};\\ 0, & i\in I_F\overset{\text{def}}{=}\{i\in I:\ F_i(x^*)=0\},\end{cases} \tag{4.3a}$$

$$(R_I^*)_{ii}=\theta(x_i^*-l_i,F_i(x^*),0)=\begin{cases}0, & i\in I_l;\\ 2, & i\in I_F,\end{cases} \tag{4.3b}$$

$$(D_J^*)_{jj}=\psi(u_j-x_j^*,-F_j(x^*),0)=\begin{cases}2, & j\in J_u\overset{\text{def}}{=}\{j\in J:\ u_j-x_j^*=0\};\\ 0, & j\in J_F\overset{\text{def}}{=}\{j\in J:\ F_j(x^*)=0\},\end{cases} \tag{4.3c}$$

$$(R_J^*)_{jj}=\theta(u_j-x_j^*,-F_j(x^*),0)=\begin{cases}0, & j\in J_u;\\ 2, & j\in J_F,\end{cases} \tag{4.3d}$$

$$(D_P^{*1})_{pp}=\psi(x_p^*-l_p,F_p(x^*)+y_p^*,0)=\begin{cases}2, & p\in P_l\overset{\text{def}}{=}\{p\in P:\ x_p^*-l_p=0\};\\ 0, & p\in P_u\overset{\text{def}}{=}\{p\in P:\ u_p-x_p^*=0\};\\ 0, & p\in P_{lu}\overset{\text{def}}{=}\{p\in P:\ l_p<x_p^*<u_p\},\end{cases} \tag{4.3e}$$

$$(R_P^{*1})_{pp}=\theta(x_p^*-l_p,F_p(x^*)+y_p^*,0)=\begin{cases}0, & p\in P_l;\\ 2, & p\in P_u;\\ 2, & p\in P_{lu},\end{cases} \tag{4.3f}$$

$$(D_P^{*2})_{pp}=\psi(u_p-x_p^*,y_p^*,0)=\begin{cases}0, & p\in P_l;\\ 2, & p\in P_u;\\ 0, & p\in P_{lu},\end{cases} \tag{4.3g}$$

$$(R_P^{*2})_{pp}=\theta(u_p-x_p^*,y_p^*,0)=\begin{cases}2, & p\in P_l;\\ 0, & p\in P_u;\\ 2, & p\in P_{lu}.\end{cases} \tag{4.3h}$$

Thus we obtain from (2.14) and the relations (4.3) above

$$(R^*)^T D^*=R^*D^*=0, \quad R_P^{*2}D_P^{*2}=0, \tag{4.4}$$

Again, we obtain from (2.14)–(2.15) and the second equation of (4.2)

$$w^T R^* G^* v=(R_P^{*1}w_P)^T D_P^{*2}v=(R_P^{*2}v)^T D_P^{*2}v=v^T(R_P^{*2}D_P^{*2})v=0. \tag{4.5}$$

So, from the first equation of (4.2), formulas (4.4), (4.5), (2.14) and taking into account the positive definition of matrix $\nabla F(x^*)$, we have

$$(R^*w)^T\nabla F(x^*)(R^*w)=0,\ R^*w=0,\ w_Q=0,\ R_P^{*1}w_P=0. \tag{4.6}$$

Furthermore we get from (4.1), (2.14)–(2.15) and (4.6)

$$0=D^*w+G^*v=\begin{pmatrix} D_I^*w_I\\ D_J^*w_J\\ D_P^{*1}w_P+D_P^{*2}v\\ 0\end{pmatrix}. \tag{4.7}$$

Adding the second equation of (4.6) into this equation and using (4.3a)–(4.3d), we have

$$(R_I^*+D_I^*)w_I=2w_I=0, \quad (R_J^*+D_J^*)w_J=2w_J=0,$$
$$w_I=0,\ w_J=0,\ (R_P^{*1}+D_P^{*1})w_P+D_P^{*2}v=0.$$

On the other hand, the second equation of (4.2) and the fourth equation of (4.6) show that

$$R_P^{*2}v=R_P^{*1}w_P=0.$$

This along with (4.3f)–(4.3h) shows that

$$v_{P_l}=0,\ v_{P_{lu}}=0,\ w_{P_u}=0,\ w_{P_{lu}}=0.$$

Finally, using $D_P^{*1}w_P+D_P^{*2}v=0$ (see (4.7)), (4.3e) and (4.3g), we easily obtain that $v_{P_u}=0$ and $w_{P_l}=0$, hence $v=0, w_P=0$ and $w=0$.

Summarizing the above discussions, we have proved that the equation system $\nabla_d\Phi(x^*,y^*,0)(w^T,v^T)^T=0$ has a unique solution zero. So the proof is completed. □

5. Implicit Continuation Algorithm

In this section, we further consider the parameter μ in (2.7) as a variable rather than a given parameter sequence $\{\mu_k\}$. So the following equation system with variable $z=(x,y,\mu)\in\Re^{n+s+1}$ is introduced.

$$\Psi(z)=\Psi(x,y,\mu)=\begin{pmatrix}\Phi(x,y,\mu)\\ e^{\mu}-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{5.1}$$

It is obvious that the function Ψ is continuously differentiable at any point (x,y,μ) with $\mu>0$, and its gradient $\nabla\Psi$ has the following expression.

$$\nabla\Psi(x,y,\mu)=\nabla_z\Psi(z)=\begin{pmatrix}\nabla_d\Phi(x,y,\mu) & 0_{(n+s)\times 1}\\ \nabla_\mu\Phi(x,y,\mu) & e^{\mu}\end{pmatrix}. \tag{5.2}$$

From Theorems 3, 4, 5 and 10 in this paper, one immediately has the following results.

THEOREM 11. (i) *The problem* (1.1) *and the equation system* (5.1) *are equivalent, i.e., x is a solution of* (1.1) *if and only if there exist a* $y\in\Re^s$ *and* $\mu\in\Re^1$ *(in fact* $\mu=0$*) such that* $z=(x,y,\mu)$ *is a solution of* (5.1). *Therefore* (5.1) *has a unique solution.*

(ii) *The gradient matrices* $\nabla\Psi(z)$ *are nonsingular at any point* $z=(x,y,\mu)$ *with* $\mu>0$.

(iii) *If Assumptions A and B hold, then the function* $\Psi(x,y,\mu)$ *is continuously differentiable at the solution* $(x^*,y^*,0)$ *of* (5.1), *and the gradient* $\nabla\Psi(x^*,y^*,0)$ *is nonsingular.*

The results above indicate that the equation system (5.1) possesses some good properties, so we can use the Newton's type methods for solving systems of nonlinear equations (see Chapter 2 in [17]) to solve (5.1), and we now present a slight modified Newton's type for (5.1) as follows.

Implicit continuation algorithm (ICA for abbreviation)

Step 0. *Choose stopping tolerances* $\delta_1,\delta_2>0$ *and a starting point* $z^0=(x^0,y^0,\mu_0)$ *with* $x^0\in X=[l,u], y^0\in\Re^s$ *and* $\mu_0>0$. *Let* $k=0$, *go to Step* 1;

Step 1. *Solve the system of linear equations*

$$A(z^k)^T \mathrm{d}z=-\Psi(z^k), \tag{5.3}$$

to obtain a solution $\mathrm{d}z^k=(\mathrm{d}x^k,\mathrm{d}y^k,\mathrm{d}\mu_k)$, *where matrix*

$$A(z^k)=\begin{pmatrix}A_d(z^k) & 0_{(n+s)\times 1}\\ A_\mu(z^k) & e^{\mu_k}\end{pmatrix}, \tag{5.4}$$

is an approximation of the gradient $\nabla\Psi(z^k)$ *in a sence such that* (5.3) *is solvable;*

Step 2. *Generate a new iterative point by* $z^{k+1} = z^k + \mathrm{d}z^k$;

Step 3. *If* $\|\mathrm{d}z^k\| \leqslant \delta_1 \|z^k\|$ *or* $\|\Psi(z^{k+1})\| \leqslant \delta_2$, *then stop and choose* z^{k+1} *and* x^{k+1} *as approximate solutions of problems* (5.2) *and* (1.1) *respectively. Otherwise, let* $k := k+1$, *go back to Step* 1.

The main properties of the ICA are summarized in the following theorem, and which can be proved easily by using directly the results on ECA or Newton's method [17].

THEOREM 12. (i) *If* $\mu > 0$ *and* $\mathrm{d}\mu$ *satisfies* $\mathrm{e}^{\mu}\mathrm{d}\mu = 1 - \mathrm{e}^{\mu}$, *then* $\mathrm{d}\mu \in (-\mu, 0)$ *and* $\mu + \mathrm{d}\mu \in (0, \mu)$, (*the proof is elementary*). *Therefore the sequence* $\{\mu_k\}$ *generated by the ICA is positive and decreasing. Furthermore, if one chooses* $A_d(z^k) = \nabla_d \Phi(z^k), A_\mu(z^k) = \nabla_\mu \Phi(z^k)$, *then* $A(z^k) = \nabla \Psi(z^k)$ *is nonsingular, therefore the system of linear equations* (5.3) *has a unique solution for all k.*

(ii) *Assume that Assumptions A and B hold. If the matrices* $A_d(z^k)$ *and* $A_\mu(z^k)$ *are computed by* $A_d(z^k) = \nabla_d \Phi(z^k), A_\mu(z^k) = \nabla_\mu \Phi(z^k)$, *and the starting point* z^0 *is located in a small neighbourhood of the solution* $z^* = (x^*, y^*, 0)$. *Then the ICA converges to* z^* *at a quadratic rate.*

6. Numerical Results

In this section, to test the efficiency of the two proposed algorithms (the ECA and the ICA), several examples have been considered. In the ECA, the error function is defined by formula (2.22), the equation system $\Phi(x, y, \mu_{k+1}) = 0$ is solved by Newton's method, and the perturbed parameter $\mu_k = (1/8)^k$. In the ICA, we compute $A_d(z^k)$ and $A_\mu(z^k)$ by $A_d(z^k) = \nabla_d \Phi(z^k)$ and $A_\mu(z^k) = \nabla_\mu \Phi(z^k)$. Our numerical tests were done at a computer with Intel CPU PI 166MHz and DOS6.22.

EXAMPLE 1. This problem is taken from [10, 13]. Let

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_2^2 + x_1 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}, \tag{6.1}$$

$$l = (0, 0, 0, 0)^T \leqslant x \leqslant u = (10, 10, 10, 10)^T.$$

the function $F(x)$ is not strongly monotone, and it has two solution points

$$x^* = (\sqrt{6}/2,\ 0,\ 0,\ 0.5)^T, \quad \bar{x}^* = (1,\ 0,\ 3,\ 0)^T,$$

however the two proposed algorithms are still effective for solving it.

EXAMPLE 2. This problem is a slight modification of the Example 2 in [17]. Take

$$F(x,z)=\begin{pmatrix} f(x)+A^T z \\ -Ax+b \end{pmatrix},$$

$$(0,0,0,0,0,0,0,0,0)^T = l \leqslant \begin{pmatrix} x \\ z \end{pmatrix} \leqslant u$$
$$= (10,5,+\infty,2,+\infty,+\infty,+\infty,+\infty,+\infty)^T,$$

where $x\in\Re^5$, $z\in\Re^4$ and

$$f(x)=\begin{pmatrix} 3.0 & -4.0 & -16.0 & -15.0 & -4.0 \\ 4.0 & 1.0 & -5.0 & -10.0 & -11.0 \\ 16.0 & 5.0 & 2.0 & -11.0 & -7.0 \\ 15.0 & 10.0 & 11.0 & 3.0 & -10.0 \\ 4.0 & 11.0 & 7.0 & 10.0 & 1.0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}+\begin{pmatrix} 0.004x_1^4 \\ 0.007x_2^4 \\ 0.005x_3^4 \\ 0.009x_4^4 \\ 0.008x_5^4 \end{pmatrix}+\begin{pmatrix} -15 \\ 10 \\ -50 \\ -30 \\ -25 \end{pmatrix},$$

$$A=\begin{pmatrix} 0.0 & 0.0 & -0.5 & 0.0 & -2.0 \\ -2.0 & -2.0 & 0.0 & -0.5 & -2.0 \\ 2.0 & 2.0 & -4.0 & 2.0 & -3.0 \\ -5.0 & 3.0 & -2.0 & 0.0 & 2.0 \end{pmatrix}, \quad b=\begin{pmatrix} -10 \\ -10 \\ 13 \\ 18 \end{pmatrix}.$$

It can be shown (see Section 2.4 in [10]) that this problem is equivalent to VIP(C,f) with the feasible set $C=\{x\in\Re^5\,|\,Ax\leqslant b, l_x\leqslant x\leqslant u_x\}$ with $l_x=(0,0,0,0,0)^T, u_x=(10,5,+\infty,2,+\infty)^T$, and VIP$(C,f)$ is a slight modification of the Example 2 in [17].

EXAMPLE 3. A Walrasian Equilibrium Model (see [3, 16]).

Consider a case with three commodities (one produced commodity and two resources), one profit maximizing producer and one utility maximizing household, which both are price takers. In particular, let the technology matrix A and the initial endowments vector b be given by

$$A=[1-1-1]^T \quad \text{and} \quad b=(0,b_2,b_3)^T, \quad \text{with } b_2>0,\ b_3>0.$$

Let the household demand functions be

$$d_i(p_1,p_2,p_3)=\frac{a_i(b_2p_2+b_3p_3)}{p_i}=\frac{a_iH}{p_i}, \quad i=1,2,3.$$

where $H=b_2p_2+b_3p_3$ denotes income. We observe that these demand functions are well defined on the interior of the price simplex $\bar{S}=\{p|p_1+p_2+p_3=1,\ p_i>0\}$. Finally, let the budget shares of household demand be $a=(a_1,a_2,a_3)=(\alpha,1-$

$\alpha, 0)$, with $0<\alpha<1$. We have chosen $b_1=0$ and $a_3=0$ in order to simplify the analysis.

In order to obtain an LCP that possibly could provide an approximate equilibrium, we have to choose a numeraire. So there are three alternative LCPs.

LCP_1:

$$F(y,p_2,p_3)=\begin{pmatrix} p_2+p_3-\bar{p}_1 \\ -y+d_{22}p_2-d_{23}p_3+b_2-d_2 \\ -y+b_3 \end{pmatrix}$$
$$l=(0,0,0)^T \leqslant (y,p_2,p_3)^T \leqslant u=(+\infty,+\infty,+\infty)^T.$$

LCP_2:

$$F(y,p_1,p_3)=\begin{pmatrix} -p_1+p_3+\bar{p}_2 \\ y+d_{11}-d_{13}-d_1-d_{12}\bar{p}_2 \\ -y+b_3 \end{pmatrix}$$
$$l=(0,0,0)^T \leqslant (y,p_1,p_3)^T \leqslant u=(+\infty,+\infty,+\infty)^T.$$

LCP_3:

$$F(y,p_1,p_2)=\begin{pmatrix} -p_1+p_2+\bar{p}_3 \\ y+d_{11}p_1-d_{12}p_2+b_2-d_2-d_1-d_{13}\bar{p}_3 \\ -y++d_{22}p_3+b_2-d_2-d_{23}\bar{p}_3 \end{pmatrix}$$
$$l=(0,0,0)^T \leqslant (y,p_1,p_2)^T \leqslant u=(+\infty,+\infty,+\infty)^T.$$

The numerical results of the ECA and the ICA for solving the three problems above with variant initial points are reported in Tables 1–10, respectively, where IP – initial point, NI – the number of iterations; APS – approximate solution; EV – error value.

Table 1. Numerical results of the ECA on Example 1

IP x^0	IP y^0	NI	APS x^*	APS y^*	EV err(x^*, y^*)
$\begin{pmatrix}1.00\\1.00\\1.00\\1.00\end{pmatrix}$	$\begin{pmatrix}1.00\\1.00\\1.00\\1.00\end{pmatrix}$	24	$\begin{pmatrix}1.22474487e+00\\1.62949468e-12\\-5.5566793e-12\\5.00000000e-01\end{pmatrix}$	$\begin{pmatrix}8.46169279e-13\\7.46920967e-13\\7.49355463e-13\\7.88498039e-13\end{pmatrix}$	2.01064521e−22
$\begin{pmatrix}1.10\\0.10\\3.10\\0.10\end{pmatrix}$	$\begin{pmatrix}1.00\\1.00\\1.00\\1.00\end{pmatrix}$	13	$\begin{pmatrix}1.00000000e+00\\4.69460481e-13\\3.00000000e+00\\3.63797338e-12\end{pmatrix}$	$\begin{pmatrix}1.61692039e-12\\1.45526580e-12\\2.07892468e-12\\1.45518648e-12\end{pmatrix}$	2.59880701e−22

Table 2. Numerical results of the ICA on Example 1

IP x^0	IP (y^0, μ_0)	NI	APS x^*	APS (y^*, μ_*)	EV $\|\Psi(z^*)\|^2$
$\begin{pmatrix}1.0\\1.0\\1.0\\1.0\end{pmatrix}$	$\begin{pmatrix}1.0\\1.0\\1.0\\1.0\\1.0\end{pmatrix}$	8	$\begin{pmatrix}1.2247434218e+00\\4.3232204412e-09\\-9.1982990627e-06\\5.0000771154e-01\end{pmatrix}$	$\begin{pmatrix}-1.3346326970e-11\\1.0904831179e-12\\-4.7897223540e-11\\4.29972651111e-11\\4.5650222062e-11\end{pmatrix}$	3.8818003754e−10
$\begin{pmatrix}1.1\\0.1\\3.1\\0.1\end{pmatrix}$	$\begin{pmatrix}1.0\\1.0\\1.0\\1.0\\1.5\end{pmatrix}$	5	$\begin{pmatrix}1.0000000066e+00\\4.3212390225e-10\\2.9999999967e+00\\3.6705416644e-09\end{pmatrix}$	$\begin{pmatrix}1.9440958374e-09\\2.0989869559e-09\\3.59470188894e-09\\2.2241317840e-09\\2.0838376113e-08\end{pmatrix}$	1.9823366913e−14

Table 3. Numerical results of the ECA on Example 2 ($\hat{x} = (x_1, x_2, x_3, x_4, x_5, z_1, z_2, z_3, z_4)^T$, $y = (y_1, y_2, y_4)^T$)

IP $\hat{x}^0$	IP y^0	NI	APS $\hat{x}^*$	APS y^*	EV $\mathrm{err}(x^*, y^*)$
1.00			9.07622922e+00		
1.00			4.84329640e+00		
1.00			2.88112641e−14		
1.00	1.00		1.70542761e−14	1.96910429e−12	
1.00	1.00	14	5.00000000e+00	1.16078344e−11	2.67484990e−22
1.00	1.00		3.72905886e+01	9.09494377e−13	
1.00			6.56348271e−14		
1.00			1.13016664e−11		
1.00			4.68209051e−14		
5.78			9.07622922e+00		
3.2363			4.84329640e+00		
654			2.86561719e−14		
1.40	564		1.84729287e−14	1.96909359e−12	
1.80	65	14	5.00000000e+00	1.16078369e−11	2.67504774e−22
765	897		3.72905886e+01	9.09494380e−13	
8.6			6.50658567e−14		
24			1.13016687e−11		
76			4.74602708e−14		

Table 4. Numerical results of the ICA on Example 2 $(\hat{x}=(x_1,x_2,x_3,x_4,x_5,z_1,z_2,z_3,z_4)^T$, $y=(y_1,y_2,y_4)^T)$

IP $\hat{x}^0$	IP (y^0,μ_0)	NI	APS $\hat{x}^*$	APS (y^*,μ_*)	EV $\|\Psi(\hat{x}^*,y^*,\mu_*)\|^2$
1.00			9.07622922e+00		
1.00			4.84329640e+00		
1.00			6.43892147e−17		
1.00	1.00		1.55707247e−15	1.60182607e−18	
1.00	1.00	12	5.00000000e+00	1.22089633e−18	5.25706715e−28
1.00	1.00		3.72905886e+01	−3.04867948e−19	
1.00	1.00		4.42539789e−16	−3.03567691e−21	
1.00			1.99425964e−18		
1.00			−3.34993192e−16		
5.78			9.07622922e+00		
3.2363			4.84329640e+00		
654			−9.95673159e−16		
1.40	564		1.10915402e−15	−7.29484300e−18	
1.80	65	17	5.00000000e+00	−4.65264345e−16	5.15581522e−28
765	897		3.72905886e+01	−3.10570379e−17	
86	10.5		1.63895269e−16	−1.81117110	
24			−4.03188655e−16		
76			4.39329347e−16		

Table 5. Numerical results of the ECA on Example 3 (LCP_1) $((b_1,b_2,b_3)=(0,100,14)$, $(\bar{p}_1,\bar{p}_2,\bar{p}_3)=(0.33,0.27,0.40)$; vector $x=(y,p_2,p_3))$

IP x^0	NI	APS x^*	EV err(x^*)
$\begin{pmatrix}11\\453\\531\end{pmatrix}$	13	$\begin{pmatrix}1.4000000000e+01\\4.7180876579e-13\\3.3000000000e-01\end{pmatrix}$	1.9457855411e−21
$\begin{pmatrix}0.34\\8756\\765\end{pmatrix}$	13	$\begin{pmatrix}1.4000000000e+01\\4.7180919947e-13\\3.3000000000e-01\end{pmatrix}$	1.9457855424e−21

Table 6. Numerical results of the ICA on Example 3 (LCP_1) $((b_1,b_2,b_3)=(0,100,14)$, $(\bar{p}_1,\bar{p}_2,\bar{p}_3)=(0.33,0.27,0.40)$; vector $x=(y,p_2,p_3))$

IP (x^0,μ_0)	NI	APS (x^*,μ_*)	EV $\|\Psi(z^*)\|^2$
$\begin{pmatrix}11\\453\\521\\11\end{pmatrix}$	19	$\begin{pmatrix}1.4000000000e+01\\-3.1921078347e-16\\3.3000000000e-01\\1.9315435295e-20\end{pmatrix}$	2.0074017681e−32
$\begin{pmatrix}0.34\\8756\\765\\7\end{pmatrix}$	14	$\begin{pmatrix}1.4000000000e+01\\-1.5632975631e-16\\3.3000000000e-01\\5.5882615594e-20\end{pmatrix}$	5.0330855092e−33

Table 7. Numerical results of the ECA on Example 3 (LCP_3) $((b_1, b_2, b_3) = (0, 51, 14),\ (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (0.33, 0.27, 0.40)$; vector $x = (y, p_1, p_3))$

IP x^0	NI	APS x^*	EV err(x^*)
$\begin{pmatrix}1.0\\1.0\\1.0\end{pmatrix}$	13	$\begin{pmatrix}2.8683501685e+00\\2.7000000000e-01\\1.3073385451e-12\end{pmatrix}$	2.9322500161e−21
$\begin{pmatrix}1561\\53\\21\end{pmatrix}$	13	$\begin{pmatrix}2.8683501685e+00\\2.7000000000e-01\\1.3072339351e-12\end{pmatrix}$	2.9322008120e−21

Table 8. Numerical results of the ICA on Example 3 (LCP_2) $((b_1, b_2, b_3) = (0, 51, 14),\ (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (0.33, 0.27, 0.40)$; vector $x = (y, p_1, p_3))$

IP (x^0, μ_0)	NI	APS (x^*, μ_*)	EV $\|\Psi(z^*)\|^3$
$\begin{pmatrix}1.0\\1.0\\1.0\\1.0\end{pmatrix}$	8	$\begin{pmatrix}2.8683501684e+00\\2.7000000000e-01\\3.273844300e-17\\-2.2317510187e-20\end{pmatrix}$	4.3001687018e−30
$\begin{pmatrix}1561\\53\\21\\11\end{pmatrix}$	17	$\begin{pmatrix}2.8683501684e+00\\2.7000000000e-01\\1.4704913546e-16\\8.4911728205e-20\end{pmatrix}$	4.8291174800e−30

Table 9. Numerical results of the ECA on Example 3 (LCP_3) $((b_1, b_2, b_3) = (0, 10, 14),\ (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (0.33, 0.27, 0.40)$; vector $x = (y, p_1, p_2))$

IP x^0	NI	APS x^*	EV err(x^*)
$\begin{pmatrix}1.0\\1.0\\1.0\end{pmatrix}$	13	$\begin{pmatrix}3.6159483626e-11\\5.0501823854e-01\\5.0745535714e-01\end{pmatrix}$	3.1614996538e−21
$\begin{pmatrix}411\\53\\41\end{pmatrix}$	13	$\begin{pmatrix}3.6159483626e-11\\5.0501823854e-01\\5.0745535714e-01\end{pmatrix}$	3.1614996538e−21

Table 10. Numerical results of the ICA on Example 3 (LCP_3) $((b_1, b_2, b_3) = (0, 10, 14),\ (\bar{p}_1, \bar{p}_2, \bar{p}_3) = (0.33, 0.27, 0.40)$; vector $x = (y, p_1, p_2))$

IP (x^0, μ_0)	NI	APS (x^*, μ_*)	EV $\|\Psi(z^*)\|^2$
$\begin{pmatrix}1.0\\1.0\\1.0\\1.0\end{pmatrix}$	14	$\begin{pmatrix}5.3992242068e-18\\5.0501823854e-01\\5.0745535714e-01\\-2.2317510187e-20\end{pmatrix}$	1.1341557229e−23
$\begin{pmatrix}411\\53\\41\\14\end{pmatrix}$	26	$\begin{pmatrix}-7.0215502463e-18\\5.0501823854e-01\\5.0745535714e-01\\3.6062379681e-20\end{pmatrix}$	1.4028287679e−24

7. Concluding Remarks

In this paper, we first have transformed the primal problem (1.1) into the equivalent system (2.1), then have introduced the perturbed complementarity problem (2.4) associated with (2.1), and have reformulated (2.4) as the equivalent nonlinear equation systems (2.7) or/and (5.1) with the help of the generalized complementarity function given by (2.5) and a special function $e^{\mu} - 1$.

Basing on the two systems of nonlinear equations (2.4) and (5.1), we have presented an explicit continuation algorithm (ECA) and an implicit continuation algorithm (ICA) for solving the primal problem (1.1), respectively. We have proved that the ECA and ICA possess satisfactory convergent properties and numerical stability under the given Assumptions A and B.

We have tested the two proposed algorithms on some practical examples, and the results have shown that the ECA and ICA are numerically effective.

We also point out that the gradients $\nabla_d \Phi(x^*, y^*, 0)$ and $\nabla_z \Psi(x^*, y^*, 0)$ may be singular and the numerical effect of the two proposed algorithms must not be stable if Assumption B does not hold. In this case, one can use some modified Newton method or quasi-Newton method to solve the system (2.4) or/and (5.1).

About the function $e^{\mu} - 1$ in the equation system (5.1), it can be replaced by another function $h(\mu)$ satisfying the conditions as follows:

(i) $h(\mu)$ is continuously differentiable and $h'(\mu) > 0$ for all $\mu \in \Re^1$;
(ii) $h(\mu) = 0$ if and only if $\mu = 0$;
(iii) If $\mu > 0$ and $d\mu$ satisfies $h(\mu) + h'(\mu) d\mu = 0$, then $\mu + d\mu > 0$.

Compared the algorithms in this paper with the existing optimizing algorithms [4, 6, 11], the latter's assumptions on the function $F(x)$ are weaker than the Assumption A stated in Section 2, but some additional assumptions are required which this paper dose not need, and the former is more simple and has smaller amount of computation, moreover, the former has more satisfactory convergent properties.

Finally, we conjecture that our algorithms can be improved such that they are suitable for the case of $F(x)$ being only monotone by perturbing $F(x)$ as $F(x) + \varepsilon x$.

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